

# The Loewner framework for model reduction of large-scale systems: An Overview and Sensitivity considerations

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Mathematics of Reduced Order Models

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# Introduction

"Model order reduction (**MOR**) is a methodology for reducing the computational complexity of mathematical models in numerical simulations. [Wikipedia]

Here (1) we will give an overview of **the data-driven modeling framework known as the Loewner framework** and (2) we will discuss issues related to the **sensitivity of the resulting models**.

## The overall problem

Physical or artificial system



PDEs



ODEs



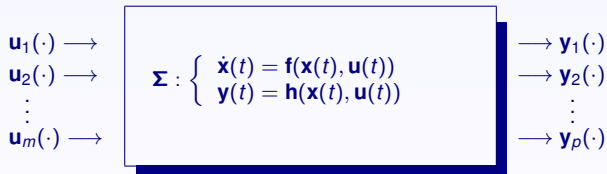
Model reduction  
(Reduced number of ODEs)

← DATA



Simulation, Design, Control

## Dynamical systems



We consider internal equations

$$\Sigma : \quad \mathbf{E} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t))$$

with **internal variable**  $\mathbf{x}(\cdot)$  of dimension  $n > m, p$ .

## Model reduction via projection

Given is 
$$\begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad \text{or} \quad \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}.$$

Common framework for (most) model reduction methods:

### Petrov-Galerkin projective approximation.

Choose  $k$ -dimensional subspaces,  $\mathcal{V}_k = \text{Range}(\mathbf{V}_k)$ ,  $\mathcal{W}_k = \text{Range}(\mathbf{W}_k) \subset \mathbb{C}^n$ .

Find  $\mathbf{v}(t) = \mathbf{V}_k \mathbf{x}_k(t) \in \mathcal{V}_k$ ,  $\mathbf{x}_k \in \mathbb{C}^k$ , such that

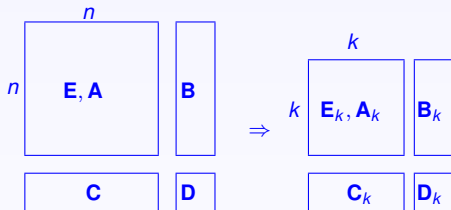
$$\begin{aligned} \mathbf{E}\dot{\mathbf{v}}(t) - \mathbf{A}\mathbf{v}(t) - \mathbf{B}\mathbf{u}(t) &\perp \mathcal{W}_k \Rightarrow \\ \mathbf{W}_k^* (\mathbf{E}\mathbf{V}_k \dot{\mathbf{x}}_k(t) - \mathbf{A}\mathbf{V}_k \mathbf{x}_k(t) - \mathbf{B}\mathbf{u}(t)) &= \mathbf{0}, \quad \mathbf{y}_k(t) = \mathbf{C}\mathbf{V}_k \mathbf{x}_k(t) + \mathbf{D}\mathbf{u}(t), \end{aligned}$$

### Reduced order system

$$\mathbf{E}_k = \mathbf{W}_k^* \mathbf{E} \mathbf{V}_k, \quad \mathbf{A}_k = \mathbf{W}_k^* \mathbf{A} \mathbf{V}_k, \quad \mathbf{B}_k = \mathbf{W}_k^* \mathbf{B}, \quad \mathbf{C}_k = \mathbf{C} \mathbf{V}_k.$$



The quality of the reduced system depends on the choice of  $\mathcal{V}_k$  and  $\mathcal{W}_k$ .



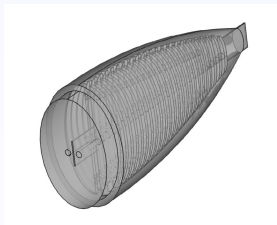
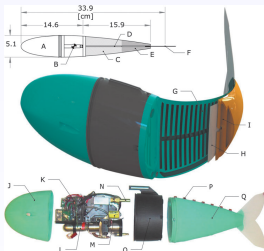
Model order reduction (MOR) seeks to reduce the complexity of large-scale dynamical systems by approximations of much lower dimension that produce nearly the same input/output response characteristics.

In the sequel we will concentrate on **interpolatory model reduction methods** and in particular on the **Loewner framework** which is **data-driven**.

## Motivating Examples

### ► Model reduction of an artificial fishtail.

- The **autonomous soft robotic fish** was developed at MIT and Worcester Polytechnic. The fish is novel because it has a soft body and has a new fluidic actuation system.



A.D. Marchese, C.D. Onal, and D. Rus, *Autonomous soft robotic fish capable of escape maneuvers using fluidic elastomer actuators*, Soft Robotics, vol. 1, pages 75-87, (2014).

- **Mathematical model.** Constitutive equations (elasticity):

$$\rho \partial_t^2 \mathbf{s}(t, \mathbf{z}) = \nabla \cdot \boldsymbol{\sigma}(\mathbf{s}(t, \mathbf{z})), \quad t > 0, \quad \mathbf{z} \in \Omega, \quad \text{where } \mathbf{s}(0, \mathbf{z}) = \mathbf{s}_0(\mathbf{z}),$$

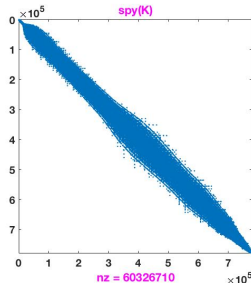
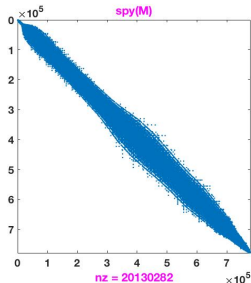
where  $\mathbf{s}(t, \mathbf{z})$  is the first-order displacement tensor,  $\boldsymbol{\sigma}(t, \mathbf{z})$  is the stress tensor and  $\Omega$  is the domain of interest.

By appropriate discretization of this partial differential equation, we obtain a second-order finite-dimensional system of the form

$$\mathbf{M} \ddot{\mathbf{x}}(t) + \mathbf{E} \dot{\mathbf{x}}(t) + \mathbf{K} \mathbf{x}(t) = \mathbf{B} \mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t),$$

where  $\mathbf{x}$  is a discretized version of  $\mathbf{s}$ , and the damping  $\mathbf{E}$  is proportional. Goal: build a controller, for the tail. The main issue at this stage is the resulting complexity, namely:

$$\mathbf{M}, \mathbf{E}, \mathbf{K} \in \mathbb{R}^{N \times N}, \quad \mathbf{B} \in \mathbb{R}^N, \quad \mathbf{C} \in \mathbb{R}^{3 \times N}, \quad \text{where } N = 779,232.$$



D. Siebelts, A. Kater, and T. Meurer, *Modeling and motion planning for an artificial fishtail*, IFAC PapersOnLine, vol. 51, pages 319-324 (2018).

## • Model order reduction

J. Saak, D. Siebelts, and S.W.R. Werner, *A comparison of second-order model order reduction methods for an artificial fishtail*, Automatisierungstechnik, vol. 67, pages 648-667 (2019).

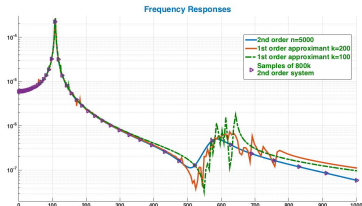
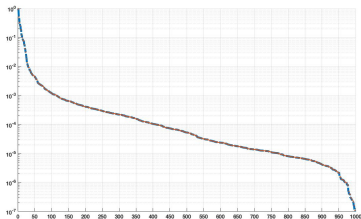
**Procedure.** Choose  $n = 500$  frequencies on the imaginary axis:  
 $\omega = \text{logspace}(-1, 3, 500)$ . Subsequently compute the 500 frequency response values:

$$\mathbf{H}(s) = \mathbf{C} \left( s^2 \mathbf{M} + s \mathbf{E} + \mathbf{K} \right)^{-1} \mathbf{B} \in \mathbb{C}^{3 \times 1}, \text{ where } s = j\omega_k, k = 1, \dots, 500.$$

**Remark:** this computation takes about 2 days.

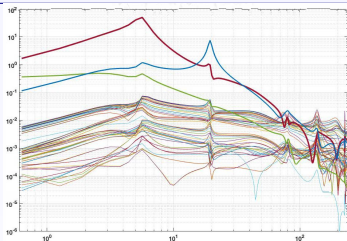
Together with the complex conjugate frequencies and transfer function values build the Loewner quadruple:

$$\mathbf{L}, \mathbf{L}_s \in \mathbb{R}^{n \times n}, \mathbf{V} \in \mathbb{R}^n, \mathbf{W} \in \mathbb{R}^{3 \times n}.$$

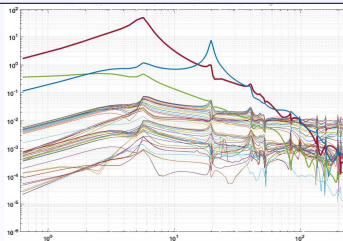


## ► Accelerating the commercial aircraft design and certification cycle.

- The design and optimization of modern commercial aircraft typically requires accurate predictions of airframe response to diverse operational loads, which may include for example, wind gusts, turbulence, wake vortices, asymmetric thrust, and vibration.
- This must be done repeatedly at different stages of the design process.
- High fidelity reduced models play a significant role in reducing overall simulation time by providing cheap and accurate surrogates for different subsystems.
- The **data-driven Loewner approach** is well suited to this purpose since the construction of reduced models can be done independently of the availability of analytical system models.
- The system input-output response was measured at 421 frequencies.
- The system input is a distributed gust disturbance.
- The system output is the lift and pitch moment at 44 locations on the wing and tail.
- The full-order system has  $5 \times 10^5$  fluid degrees of freedom and  $2 \times 10^3$  structural degrees of freedom.
- The reduced dynamical system has 30 degrees of freedom.
- The relative deviation between the reduced system response and the full-order system response is never larger than 0.35%



**Original (full-order) frequency response**



**Reduced (order 30) frequency response**



**Matlab Loewner Toolbox  
C. Poussot-Vassal & P. Vuillemin**



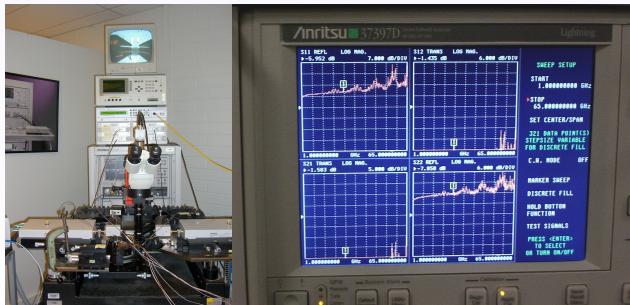
**Dassault business jet test**

## ► Microstrip device.



Microstrip transmission lines are a common way to connect two devices. They consist of two thin parallel (metal) strips, mounted on the same dielectric substrate.

Microstrip devices are described by **S-parameters** (Scattering parameters).



VNA (Vector Network Analyzer) and VNA screen showing the magnitude of the S-parameters for a 2 port device.

# The Loewner matrix

Given:

row array  $(\mu_j, \mathbf{v}_j)$ ,  $j = 1, \dots, q$ ,

column array  $(\lambda_i, \mathbf{w}_i)$ ,  $i = 1, \dots, k$ ,

the associated **Loewner matrix** is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1 - \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q - \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \frac{\mathbf{v}_q - \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix} \in \mathbb{C}^{q \times k}$$

If  $\mathbf{w}_i = \mathbf{g}(\lambda_i)$ ,  $\mathbf{v}_j = \mathbf{g}(\mu_j)$ , are **samples** of  $\mathbf{g}$ :

**Main property.** Let  $\mathbb{L}$  be as above.

Then  $k, q \geq \deg \mathbf{g} \Rightarrow \text{rank } \mathbb{L} = \deg \mathbf{g}$ .

## Karel Löwner (1893 - 1968)



Ch. Loewner

- Born in Bohemia
- Studied in Prague under Georg Pick
- Emigrated to the US in 1939
- Seminal paper:  
Über monotone Matrixfunktionen,  
Math. Zeitschrift (1934).



## Rational interpolation and the Loewner matrix

- **Lagrange basis** for space of polynomials of degree at most  $n$ .

Given distinct  $\lambda_i \in \mathbb{C}$ ,  $i = 1, \dots, n+1$ , define

$$\mathbf{q}_i(s) := \prod_{k \neq i} (s - \lambda_k), \quad i = 1, \dots, n+1.$$

- For given constants  $\alpha_i, \mathbf{w}_i$ ,  $i = 1, \dots, n+1$ , consider

$$\sum_{i=1}^{n+1} \alpha_i \frac{\mathbf{g} - \mathbf{w}_i}{s - \lambda_i} = 0, \quad \alpha_i \neq 0.$$

- Solving for  $\mathbf{g}$  we obtain

$$\boxed{\mathbf{g}(s) = \frac{\sum_{i=1}^{n+1} \frac{\alpha_i \mathbf{w}_i}{s - \lambda_i}}{\sum_{i=1}^{n+1} \frac{\alpha_i}{s - \lambda_i}}} \Rightarrow \mathbf{g}(\lambda_i) = \mathbf{w}_i.$$

The free parameters  $\alpha_i$ , can be specified so that

$$\mathbf{g}(\mu_j) = \mathbf{v}_j, \quad j = 1, \dots, r,$$

where  $(\mu_j, \mathbf{v}_j)$ ,  $\mu_i \neq \mu_j$ , are given. For this to hold  $\mathbb{L}\mathbf{c} = 0$ , where

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1 - \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1 - \mathbf{w}_{n+1}}{\mu_1 - \lambda_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_r - \mathbf{w}_1}{\mu_r - \lambda_1} & \dots & \frac{\mathbf{v}_r - \mathbf{w}_{n+1}}{\mu_r - \lambda_{n+1}} \end{bmatrix} \in \mathbb{C}^{r \times (n+1)}, \quad \mathbf{c} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{n+1} \end{bmatrix} \in \mathbb{C}^{n+1}.$$

$\mathbb{L}$ : **Loewner matrix** with

**row array:**  $(\mu_j, \mathbf{v}_j), \quad j = 1, \dots, r, \quad \text{and}$

**column array:**  $(\lambda_i, \mathbf{w}_i), \quad i = 1, \dots, n+1.$

## Main property

Let  $\mathbb{L}$  be a  $p \times k$  Loewner matrix with  $p, k \geq \deg \mathbf{g}$ . Then

$\text{rank } \mathbb{L} = \deg \mathbf{g}.$

- A.C. Antoulas and B.D.O. Anderson, *On the scalar rational interpolation problem*, IMA Journal of Mathematical Control & Information, vol. 3, pages 61-88 (1986).

Y. Nakatsukasa, O. Sete, and L.N. Trefethen, The AAA algorithm for rational interpolation.

## THE AAA ALGORITHM FOR RATIONAL APPROXIMATION

YUJI NAKATSUKASA\*, OLIVIER SÈTE†, AND LLOYD N. TREFETHEN‡

*For Jean-Paul Berrut, the pioneer of numerical algorithms based on rational barycentric representations, on his 65th birthday.*

**Abstract.** We introduce a new algorithm for approximation by rational functions on a real interval or a set in the complex plane, implementable in 40 lines of Matlab. Even on a disk or interval the algorithm may outperform existing methods, and on more complicated domains it is especially competitive. The core ideas are (1) representation of the rational approximant in barycentric form with interpolation at certain support points, (2) greedy selection of the support points to avoid exponential instabilities, and (3) least-squares rather than interpolatory formulation of the overall problem. The name AAA stands for “aggressive Antoulas–Anderson” in honor of the authors who introduced a scheme based on (1). We present the core algorithm with a Matlab code and eight applications and describe variants targeted at problems of different kinds.

**Key words.** rational approximation, barycentric formula, analytic continuation, AAA algorithm, Froissart doublet

**AMS subject classifications.** 41A20, 65D15

Revised version: aggressive → adaptive

# The Loewner framework

Given: **right data**:  $(\lambda_i; \mathbf{r}_i, \mathbf{w}_i)$ ,  $i = 1, \dots, k$ , and **left data**:  $(\mu_j; \ell_j^*, \mathbf{v}_j^*)$ ,  $j = 1, \dots, q$  (for simplicity all points are assumed distinct).

**Problem**: Find rational  $p \times m$  matrices  $\mathbf{H}(s)$ , such that:  $\boxed{\mathbf{H}(\lambda_i) \mathbf{r}_i = \mathbf{w}_i}$ ,  $\boxed{\ell_j^* \mathbf{H}(\mu_j) = \mathbf{v}_j^*}$ .

**Right data**:

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \in \mathbb{C}^{k \times k}, \quad \mathbf{R} = [\mathbf{r}_1 \ \mathbf{r}_2 \ \cdots \ \mathbf{r}_k] \in \mathbb{C}^{m \times k},$$
$$\mathbf{W} = [\mathbf{w}_1 \ \mathbf{w}_2 \ \cdots \ \mathbf{w}_k] \in \mathbb{C}^{p \times k},$$

**Left data**:

$$M = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_q \end{bmatrix} \in \mathbb{C}^{q \times q}, \quad \mathbf{L} = \begin{bmatrix} \ell_1^* \\ \vdots \\ \ell_q^* \end{bmatrix} \in \mathbb{C}^{q \times p}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_q^* \end{bmatrix} \in \mathbb{C}^{q \times m}$$

## State-space representation: the Loewner matrix pair

Recall data:  $\mathbf{H}(\lambda_i)\mathbf{r}_i = \mathbf{w}_i$ ,  $\ell_j^*\mathbf{H}(\mu_j) = \mathbf{v}_j^*$ .

The **Loewner matrix**  $\mathbb{L} \in \mathbb{C}^{q \times k}$  is:

$$\mathbb{L} = \begin{bmatrix} \frac{\mathbf{v}_1^* \mathbf{r}_1 - \ell_1^* \mathbf{w}_1}{\mu_1 - \lambda_1} & \dots & \frac{\mathbf{v}_1^* \mathbf{r}_k - \ell_1^* \mathbf{w}_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mathbf{v}_q^* \mathbf{r}_1 - \ell_q^* \mathbf{w}_1}{\mu_q - \lambda_1} & \dots & \frac{\mathbf{v}_q^* \mathbf{r}_k - \ell_q^* \mathbf{w}_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}$  satisfies the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = \mathbf{V}\mathbf{R} - \mathbf{L}\mathbf{W}$$

If  $\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$ , is given:

$$\mathbf{x}_i := (\lambda_i \mathbf{E} - \mathbf{A})^{-1} \mathbf{B} \mathbf{r}_i, \quad \mathbf{y}_j := \ell_j^* \mathbf{C} (\mu_j \mathbf{E} - \mathbf{A})^{-1}$$

$\Rightarrow$

**X**, **Y**: gen. reach./observ. matrix  $\Rightarrow$

$$\Rightarrow \mathbb{L} = -\mathbf{Y}\mathbf{X}$$

The **shifted Loewner matrix**  $\mathbb{L}_S \in \mathbb{C}^{q \times k}$  is:

$$\mathbb{L}_S = \begin{bmatrix} \frac{\mu_1 \mathbf{v}_1^* \mathbf{r}_1 - \ell_1^* \mathbf{w}_1 \lambda_1}{\mu_1 - \lambda_1} & \dots & \frac{\mu_1 \mathbf{v}_1^* \mathbf{r}_k - \ell_1^* \mathbf{w}_k \lambda_k}{\mu_1 - \lambda_k} \\ \vdots & \ddots & \vdots \\ \frac{\mu_q \mathbf{v}_q^* \mathbf{r}_1 - \ell_q^* \mathbf{w}_1 \lambda_1}{\mu_q - \lambda_1} & \dots & \frac{\mu_q \mathbf{v}_q^* \mathbf{r}_k - \ell_q^* \mathbf{w}_k \lambda_k}{\mu_q - \lambda_k} \end{bmatrix}$$

$\mathbb{L}_S$  satisfies the Sylvester equation

$$\mathbb{L}_S \Lambda - M \mathbb{L}_S = \mathbf{M} \mathbf{V} \mathbf{R} - \mathbf{L} \mathbf{W} \Lambda$$

$\mathbb{L}_S$  can be factored as

$$\Rightarrow \mathbb{L}_S = -\mathbf{Y} \mathbf{A} \mathbf{X}$$

## Construction of Models or **Learning** from the data

- If  $\det(x\mathbb{L} - \mathbb{L}_s) \neq 0$ ,  $x \in \{\lambda_i\} \cup \{\mu_j\}$ , then

$$\mathbf{E} = -\mathbb{L}, \quad \mathbf{A} = -\mathbb{L}_s, \quad \mathbf{B} = \mathbf{V}, \quad \mathbf{C} = \mathbf{W}$$

is a minimal interpolant of the data, i.e.,  $\mathbf{H}(s)$  interpolates the data:

$$\mathbf{H}(s) = \mathbf{W}(\mathbb{L}_s - s\mathbb{L})^{-1}\mathbf{V}$$

- Otherwise, if the **numerical**  $\text{rank } \mathbb{L} = k$ , compute the rank revealing SVD:

$$\mathbb{L} = \mathbf{Y}\Sigma\mathbf{X}^* \approx \mathbf{Y}_k\Sigma_k\mathbf{X}_k^*$$

**Theorem.** A realization  $[\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C}]$ , of an approximate interpolant is given as follows:

$$\mathbf{E} = -\mathbf{Y}_k^*\mathbb{L}\mathbf{X}_k, \quad \mathbf{A} = -\mathbf{Y}_k^*\mathbb{L}_s\mathbf{X}_k, \quad \mathbf{B} = \mathbf{Y}_k^*\mathbf{V}, \quad \mathbf{C} = \mathbf{W}\mathbf{X}_k.$$

## The Loewner Algorithm (simple version)

1. Consider given (frequency domain) measurements  $(s_i, \phi_i)$ ,  $i = 1, \dots, N$ .
2. Partition the measurements into 2 disjoint sets

$$\text{frequencies : } [s_1, \dots, s_N] = [\lambda_1, \dots, \lambda_k], [\mu_1, \dots, \mu_q], \quad k + q = N,$$

$$\text{values : } [\phi_1, \dots, \phi_N] = [w_1, \dots, w_k], [v_1, \dots, v_q] = \mathbf{W}, \mathbf{V}^T.$$

3. Construct the **Loewner pencil**:

$$\mathbb{L} = \left( \frac{v_i - w_j}{\mu_i - \lambda_j} \right)_{i=1, \dots, q}^{j=1, \dots, k}, \quad \mathbb{L}_s = \left( \frac{\mu_i v_i - \lambda_j w_j}{\mu_i - \lambda_j} \right)_{i=1, \dots, q}^{j=1, \dots, k}.$$

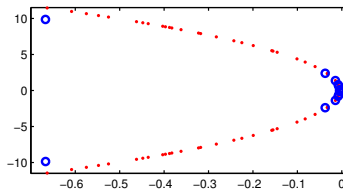
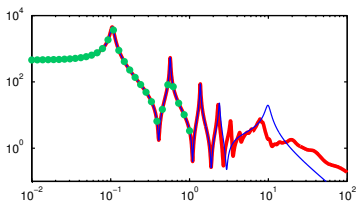
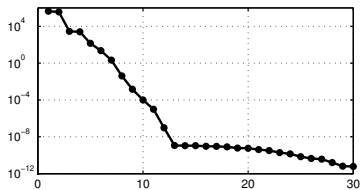
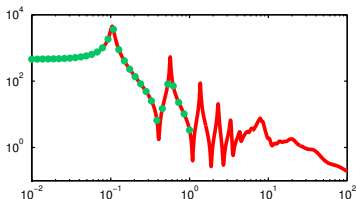
4. It follows that the **raw model** is:  $(\mathbf{W}, \mathbb{L}, \mathbb{L}_s, \mathbf{V})$ .
5. Compute the rank revealing SVD:  $\mathbb{L} \approx \mathbf{Y} \Sigma \mathbf{X}^*$  ( $\Sigma \in \mathbb{R}^{r \times r}$ ).
6. The reduced model  $(\hat{\mathbf{C}}, \hat{\mathbf{E}}, \hat{\mathbf{A}}, \hat{\mathbf{B}})$  is obtained by **projecting** the raw model  $(\mathbf{W}, \mathbb{L}, \mathbb{L}_s, \mathbf{V})$ :

$$\hat{\mathbf{C}} = \mathbf{W} \mathbf{X}, \quad \hat{\mathbf{E}} = -\mathbf{Y}^* \mathbb{L} \mathbf{X}, \quad \hat{\mathbf{A}} = -\mathbf{Y}^* \mathbb{L}_s \mathbf{X}, \quad \hat{\mathbf{B}} = \mathbf{Y}^* \mathbf{V}.$$

7. **Reference**: S. Lefteriu and A.C. Antoulas: A New Approach to Modeling Multiport Systems from Frequency-Domain Data, IEEE Trans. CAD, 29: 14-27 (2010).

## Example: a discretized Euler-Bernoulli beam

- System of order  $n = 348$  (obtained after discretization) representing a clamped beam.
- $N = 60$  frequency response measurements,  $s_k = j\omega_k$ , with  $\omega_k \in [-1, -0.01] \cup [0.01, 1]$ .
- Construct  $30 \times 30$  Loewner pencil and  $\mathbf{Y}, \mathbf{X} \in \mathbb{R}^{30 \times 12}$  from the SVD.
- Project to get reduced model of order  $r = 12$ .



(1,1) Original and data

(1,2) Singular values of  $\mathbb{L}$ .

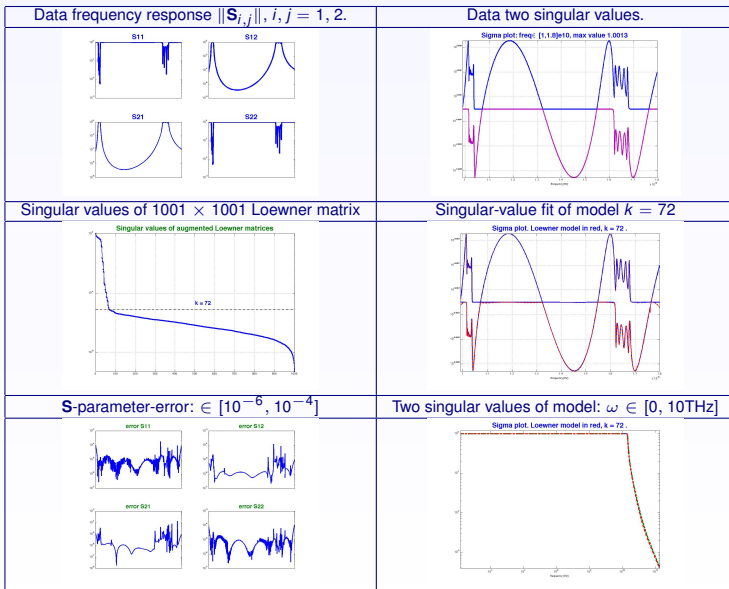
(2,1) Original & reduced FR

(2,2) Poles original & reduced



# Reduced model from frequency response measurements

1001 S-parameter measurements between 10-18 GHz (CST).



## The issue of sensitivity

### A general factorization of the Loewner pencil and its consequences.

Given the system  $\Sigma$  and a minimal realization thereof:

$$\Sigma = (\mathbf{C}, \mathbf{E}, \mathbf{A}, \mathbf{B}) \in \mathbb{R}^{p \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m},$$

let the associated resolvent be denoted by  $\Phi(s) = (s\mathbf{E} - \mathbf{A})^{-1}$ , and the transfer function by  $\mathbf{H}(s) = \mathbf{C}\Phi(s)\mathbf{B}$ . Next, consider the Loewner data below and the resulting Loewner quadruple:

$$\underbrace{(\mathbf{M}, \mathbf{L}^T, \mathbf{V}^T)}_{\text{left data: } q \times q, q \times p, q \times m} \quad \text{and} \quad \underbrace{(\mathbf{\Lambda}, \mathbf{R}, \mathbf{W})}_{\text{right data: } k \times k, m \times k, p \times k} \quad \Rightarrow \quad \underbrace{(\mathbf{W}, \mathbf{L}, \mathbf{L}_s, \mathbf{V}^T)}_{\text{Loewner quadruple: } p \times k, q \times k, q \times k, q \times m}$$

Let  $\mathcal{C}_L$  and  $\mathcal{C}_R$  be the associated left/right **rational Krylov** projectors:

$$\mathcal{C}_L = \begin{bmatrix} \ell_1^T \mathbf{C} \Phi(\mu_1) \\ \vdots \\ \ell_q^T \mathbf{C} \Phi(\mu_q) \end{bmatrix} \in \mathbb{C}^{q \times n}, \quad \mathcal{C}_R = [ \mathbf{\Phi}(\lambda_1) \mathbf{B} \mathbf{r}_1 \quad \cdots \quad \mathbf{\Phi}(\lambda_k) \mathbf{B} \mathbf{r}_k ] \in \mathbb{C}^{n \times k},$$

assumed to satisfy the condition that  $\mathcal{C}_L \mathcal{C}_R$  has full rank.

It readily follows that the following Sylvester equations are satisfied:

$$\mathbf{M} \mathbf{C}_L \mathbf{E} - \mathbf{C}_L \mathbf{A} = \mathbf{L}^T \mathbf{C} \quad \text{and} \quad \mathbf{E} \mathbf{C}_R \mathbf{\Lambda} - \mathbf{A} \mathbf{C}_R = \mathbf{B} \mathbf{R}$$

Multiplying the **former equation** by  $\mathbf{C}_R$  on the right and the latter by  $\mathbf{C}_L$  on the left we obtain

$$\underbrace{\mathbf{M} \mathbf{C}_L \mathbf{E} \mathbf{C}_R}_{\mathbf{L}} - \underbrace{\mathbf{C}_L \mathbf{A} \mathbf{C}_R}_{\mathbf{L}_S} = \mathbf{L}^T \cdot \underbrace{\mathbf{C} \mathbf{C}_R}_{\mathbf{W}} \quad \text{and} \quad \underbrace{\mathbf{C}_L \mathbf{E} \mathbf{C}_R \mathbf{\Lambda}}_{\mathbf{L}} - \underbrace{\mathbf{C}_L \mathbf{A} \mathbf{C}_R}_{\mathbf{L}_S} = \underbrace{\mathbf{C}_L \mathbf{B} \cdot \mathbf{R}}_{\mathbf{V}}$$

which yield the well known relationships:  $\mathbf{M} \mathbf{L} - \mathbf{L}_S = \mathbf{L} \mathbf{W}$  and  $\mathbf{L} \mathbf{\Lambda} - \mathbf{L}_S = \mathbf{V} \mathbf{R}$ .

Appropriate addition/subtraction of these equations yields in turn the Sylvester equations satisfied by the Loewner pencil:

$$\mathbf{L} \mathbf{\Lambda} - \mathbf{M} \mathbf{L} = \mathbf{V} \mathbf{R} - \mathbf{L} \mathbf{W} \quad \text{and} \quad \mathbf{L}_S \mathbf{\Lambda} - \mathbf{M} \mathbf{L}_S = (\mathbf{M} \mathbf{V}) \mathbf{R} - \mathbf{L} (\mathbf{W} \mathbf{\Lambda}).$$

Thus the following factorizations hold:

$$\mathbf{L} = \mathbf{C}_L \mathbf{E} \mathbf{C}_R, \quad \mathbf{L}_S = \mathbf{C}_L \mathbf{A} \mathbf{C}_R, \quad \mathbf{W} = \mathbf{C} \mathbf{C}_R, \quad \mathbf{V} = \mathbf{C}_L \mathbf{B}$$

**Lemma.** The above factorizations lead to three conclusions.

- ▶ **(a)**  $(\mathbb{L}_S, \mathbb{L})$  is the Loewner pencil, i.e.  $(\mathbb{L})_{i,j} = \frac{\mathbf{v}_i^T \mathbf{r}_j - \ell_i^T \mathbf{w}_j}{\lambda_j - \mu_i}$ ,  $(\mathbb{L}_S)_{i,j} = \frac{\mu_i \mathbf{v}_i^T \mathbf{r}_j - \ell_i^T \mathbf{w}_j \lambda_j}{\lambda_j - \mu_i}$ .

Thus the rank of  $\mathbb{L}$  is equal to the McMillan degree  $n$  of the system  $\Sigma$ .

- ▶ **(b)** The factorizations are rank revealing.
- ▶ **(c)** An explicit generalized EVD (eigenvalue decomposition) of the Loewner pencil  $(\mathbb{L}_S, \mathbb{L})$  results, namely

$$\hat{\mathbf{v}} = C_R^+ \mathbf{v} \quad \text{and} \quad \hat{\mathbf{w}}^T = \mathbf{w}^T C_L^+.$$

**Proof.** We will show part **(c)**. Let  $(\lambda, \mathbf{v}, \mathbf{w})$  be a right/left eigenvalue/eigenvector triple of  $(\mathbf{A}, \mathbf{E})$ . Then  $\hat{\mathbf{v}} = C_R^+ \mathbf{v}$ , where the exponent  $+$  denotes the Moore-Penrose pseudo inverse, is the right eigenvector of the Loewner pencil  $(\mathbb{L}_S, \mathbb{L})$ , corresponding to the eigenvalue  $\lambda$ :

$$\mathbb{L}_S \hat{\mathbf{v}} = C_L \mathbf{A} C_R \hat{\mathbf{v}} = C_L \mathbf{A} C_R C_R^+ \mathbf{v} = C_L \mathbf{A} \mathbf{v} = \lambda C_L \mathbf{E} \mathbf{v} = \lambda C_L \mathbf{E} C_R C_R^+ \mathbf{v} = \lambda C_L \mathbf{E} C_R \hat{\mathbf{v}} = \lambda \mathbb{L} \hat{\mathbf{v}}.$$

Similarly for the left eigenvectors associated with the eigenvalues of  $(\mathbb{L}_S, \mathbb{L})$ . ■

**Remark.** The fact that rational Krylov projections are associated with the Loewner pencil, brings two issues into the picture: (1) the fact that the **rank of  $\mathbb{L}$**  is equal to the **McMillan degree** of the underlying system and (2) the fact that an **explicit GEVD** (generalized eigenvalue decomposition) of the projected pencil results. ■

## Basics of perturbation theory

Consider the generalized eigenvalue problem of the pencil  $(\mathbf{A}, \mathbf{E})$ :

$$\mathbf{A}\mathbf{v} = \sigma\mathbf{E}\mathbf{v}, \quad \mathbf{w}^T\mathbf{A} = \sigma\mathbf{w}^T\mathbf{E},$$

where  $\sigma$  is assumed to be a simple eigenvalue. An  $\epsilon$ -perturbation of the system yields

$$\begin{aligned}(\mathbf{A} + \epsilon\Delta_{\mathbf{A}})(\mathbf{v} + \epsilon\mathbf{v}^{(1)} + \dots) &= (\sigma + \epsilon\sigma^{(1)} + \dots)(\mathbf{E} + \epsilon\Delta_{\mathbf{E}})(\mathbf{v} + \epsilon\mathbf{v}^{(1)} + \dots), \\(\mathbf{w}^T + \epsilon\mathbf{w}^{(1)T} + \dots)(\mathbf{A} + \epsilon\Delta_{\mathbf{A}}) &= (\sigma + \epsilon\sigma^{(1)} + \dots)(\mathbf{w}^T + \epsilon\mathbf{w}^{(1)T} + \dots)(\mathbf{E} + \epsilon\Delta_{\mathbf{E}}).\end{aligned}$$

Retaining only the first-order terms yields  $(\mathbf{A} - \sigma\mathbf{E})\mathbf{v}^{(1)} = (\sigma^{(1)}\mathbf{E} + \sigma\Delta_{\mathbf{E}} - \Delta_{\mathbf{A}})\mathbf{v}$ . We wish to find an expression for  $\sigma^{(1)}$ , which measures the first-order sensitivity of  $\sigma$ . Towards this goal we multiply this equation on the left by the corresponding left eigenvector  $\mathbf{w}$ :

$$\begin{aligned}\mathbf{w}^T(\mathbf{A} - \sigma\mathbf{E})\mathbf{v}^{(1)} &= \mathbf{w}^T(\sigma^{(1)}\mathbf{E} + \sigma\Delta_{\mathbf{E}} - \Delta_{\mathbf{A}})\mathbf{v} \Rightarrow \\ \sigma^{(1)} &= \frac{\mathbf{w}^T\Delta_{\mathbf{A}}\mathbf{v} - \sigma\mathbf{w}^T\Delta_{\mathbf{E}}\mathbf{v}}{\mathbf{w}^T\mathbf{E}\mathbf{v}} = \frac{\mathbf{w}^T(\Delta_{\mathbf{A}} - \sigma\Delta_{\mathbf{E}})\mathbf{v}}{\mathbf{w}^T\mathbf{E}\mathbf{v}}.\end{aligned}$$

If we now assume that  $\|\Delta_{\mathbf{A}}\| = \|\mathbf{A}\|$ ,  $\|\Delta_{\mathbf{E}}\| = \|\mathbf{E}\|$ , we obtain

$$|\sigma^{(1)}| \leq \frac{\|\mathbf{w}\| (\|\mathbf{A}\| + |\sigma|\|\mathbf{E}\|) \|\mathbf{v}\|}{|\mathbf{w}^T\mathbf{E}\mathbf{v}|} = \text{const}_{\sigma} \cdot \underbrace{\frac{\|\mathbf{w}\| \cdot \|\mathbf{v}\|}{|\mathbf{w}^T\mathbf{E}\mathbf{v}|}}_{\text{sens}_{\sigma}}.$$

If  $\mathbf{E} = \mathbf{I}$ , we obtain the well known result:  $\frac{1}{\text{sens}_{\sigma}} = \cos \angle(\mathbf{v}, \mathbf{w})$ .

## The SISO case

Let an underlying linear siso ( $m = p = 1$ ) system of dimension  $n$  be represented by means of its partial fraction decomposition:

$$\mathbf{H}(s) = \frac{\mathbf{n}(s)}{\mathbf{d}(s)} = \sum_{i=1}^n \frac{\gamma_i}{s - \pi_i},$$

where  $0 \neq \gamma_i \in \mathbb{C}$ , for all  $i$ .

To this system we associate a Loewner quadruple  $(\mathbb{W}, \mathbb{L}_s, \mathbb{L}, \mathbb{V})$  constructed by means of the left interpolation points  $\mu_i, i = 1, \dots, q$ , the right interpolation points  $\lambda_j, j = 1, \dots, k$  (assumed distinct), the left values  $\mathbb{V}_i = \mathbf{H}(\mu_i)$  and the right values  $\mathbb{W}_j = \mathbf{H}(\lambda_j)$ , where  $q, k \geq n$ . With  $\mathbf{R} = \mathbf{1}_q^T, \mathbf{L} = \mathbf{1}_k$ , the Loewner pencil satisfies the Sylvester equations:

$$\mathbb{L}\mathbf{A} - \mathbf{M}\mathbb{L} = \mathbb{V}\mathbf{R} - \mathbf{L}\mathbb{W}, \quad \mathbb{L}_s\mathbf{A} - \mathbf{M}\mathbb{L}_s = (\mathbf{M}\mathbb{V})\mathbf{R} - \mathbf{L}(\mathbb{W}\mathbf{A}).$$

In the sequel we will denote the poles of the system by:  $\mathbf{\Pi} = \text{diag} [\pi_1, \dots, \pi_n] \in \mathbb{C}^{n \times n}$ .

Given two sets of mutually distinct complex numbers  $\alpha_i, i = 1, \dots, \kappa$ , and  $\beta_j, j = 1, \dots, \rho$ , we define the associated **Cauchy matrix**:

$$\mathcal{C}_{\alpha, \beta} = \begin{bmatrix} \frac{1}{\alpha_1 - \beta_1} & \cdots & \frac{1}{\alpha_1 - \beta_\rho} \\ \vdots & \cdots & \vdots \\ \frac{1}{\alpha_\kappa - \beta_1} & \cdots & \frac{1}{\alpha_\kappa - \beta_\rho} \end{bmatrix} \in \mathbb{C}^{\kappa \times \rho}.$$

This Cauchy matrix satisfies the Sylvester equation:

$$\mathcal{C}_{\alpha, \beta} \mathbf{A} - \mathbf{B} \mathcal{C}_{\alpha, \beta} = \mathbf{1}_\kappa \mathbf{1}_\rho^T, \quad \text{where } \mathbf{A} = \text{diag} [\alpha_1, \dots, \alpha_\kappa], \quad \mathbf{B} = \text{diag} [\beta_1, \dots, \beta_\rho].$$

**Lemma.** In the above setting, the relationships below hold true, where the matrix containing the residues  $\gamma_i$  is denoted by:  $\mathbf{\Gamma} = \text{diag} [\gamma_1, \dots, \gamma_n]$ :

$\mathbf{W}$	$=$	$\mathbf{1}_n^T \mathbf{\Gamma} \mathbf{C}_{\pi, \lambda}$	$\in \mathbb{C}^{1 \times k},$
$\mathbf{L}$	$=$	$\mathbf{C}_{\mu, \pi} \mathbf{\Gamma} \mathbf{C}_{\pi, \lambda}$	$\in \mathbb{C}^{q \times k},$
$\mathbf{L}_S$	$=$	$\mathbf{C}_{\mu, \pi} \mathbf{\Pi} \mathbf{\Gamma} \mathbf{C}_{\pi, \lambda}$	$\in \mathbb{C}^{q \times k},$
$\mathbf{V}$	$=$	$\mathbf{C}_{\mu, \pi} \mathbf{\Gamma} \mathbf{1}_n$	$\in \mathbb{C}^{q \times 1}.$

- P. Amstutz, Une méthode d'interpolation par les fonctions rationnelles, Annales des Télécommunications, tome 22, Mars-Avril 1967, pages 62-54.

The proof is based on the fact that because the simultaneous factorization of the Loewner pencil is rank revealing, the **Moore-Penrose generalized (or pseudo-) inverse** of the (in general) rectangular matrix

$$\Phi(s) = s\mathbb{L} - \mathbb{L}_s = \underbrace{C_{\mu,\pi}}_{q \times n} \underbrace{[s\Gamma - \Gamma\Pi]}_{n \times n} \underbrace{C_{\pi,\lambda}}_{n \times k} \in \mathbb{C}^{q \times k},$$

is as follows:

$$\Phi^{\text{MP}}(s) = C_{\pi,\lambda}^* \left( C_{\pi,\lambda} C_{\pi,\lambda}^* \right)^{-1} [s\Gamma - \Gamma\Pi]^{-1} (C_{\mu,\pi}^* C_{\mu,\pi})^{-1} C_{\mu,\pi}^* \in \mathbb{C}^{k \times q} \Rightarrow$$

$$\mathbb{W} \Phi^{\text{MP}}(s) \mathbb{V} = [\gamma_1, \dots, \gamma_n] (\text{diag} [s - \pi_1, \dots, s - \pi_n])^{-1} \mathbf{1}_n = \sum_{i=1}^n \frac{\gamma_i}{s - \pi_i} = \mathbf{H}(s).$$

Thus, although the Loewner pencil is rectangular and/or singular, using the Moore-Penrose pseudo-inverse, we recover the original system, **without** the need of an explicit **projection**.



### The sensitivity formula.

From the previous section we can now insert the expressions for  $\mathbf{v}$  and  $\mathbf{w}$  as a function of the Cauchy matrices  $\mathcal{C}_L = \mathcal{C}_{\mu,\pi}$  and  $\mathcal{C}_R = \mathcal{C}_{\pi,\lambda}$ . In addition, if the original system realization is as above, the sensitivity of the  $i^{\text{th}}$  pole, is up to a constant, given by

$$\text{sens}_i = \frac{\|\mathbf{e}_i^T \mathcal{C}_{\mu,\pi}^+\| \cdot \|\mathcal{C}_{\pi,\lambda}^+ \mathbf{e}_i\|}{|\mathbf{e}_i^T \mathcal{C}_{\mu,\pi}^+ \mathbb{L} \mathcal{C}_{\pi,\lambda}^+ \mathbf{e}_i|} = \frac{\|\mathbf{e}_i^T \mathcal{C}_{\mu,\pi}^+\| \cdot \|\mathcal{C}_{\pi,\lambda}^+ \mathbf{e}_i\|}{|\gamma_i|}$$

We notice that this expression depends on the interpolation points chosen, through the **Cauchy matrices**, as well as the on the size of the corresponding **residue**; hence the condition numbers of these matrices and the residue are relevant for determining  $\text{sens}_i$ .

## Consequences: eigenvalue sensitivity

1. **GEVD of the Loewner pencil.** The right eigenvector of the Loewner Matrix Pencil  $(\mathbb{L}_s, \mathbb{L})$  corresponding to the eigenvalue  $\pi_r$  is

$$q_r = (C_R)^+ e_r,$$

where  $+$  is the notation of pseudo-inverse and  $e_r$  is the unit vector whose  $r^{th}$  entry is 1 while the rest are 0. Similarly, the left eigenvector corresponding to the same eigenvalue is

$$p_r = (C_L^T)^+ e_r.$$

If the system is SISO, then the eigenvector of Loewner matrix pencil can be also obtained by

$$q_r = (C_{\lambda, \pi}^T)^+ e_r, \quad p_r = (C_{\mu, \pi}^T)^+ e_r.$$

Here the difference of the two expressions is the norm of the eigenvector.

2. **Sensitivity of the poles for Loewner pencil.** For any matrix pencil  $(\mathbf{A}, \mathbf{E})$ , under the perturbation of  $\bar{\mathbf{A}} = \mathbf{A} + \mathbf{N}_A$  and  $\bar{\mathbf{E}} = \mathbf{E} + \mathbf{N}_E$ , the first order approximation of the eigenvalue perturbation  $\delta\pi$  is

$$\delta\pi = \frac{p^T (\mathbf{N}_A - \pi \mathbf{N}_E) q}{p^T \mathbf{E} q}.$$

Similarly, for the Loewner pencil if the perturbed matrices are  $\bar{\mathbb{L}} = \mathbb{L} + \mathbf{N}$ ,  $\bar{\mathbb{L}}_s = \mathbb{L}_s + \mathbf{N}_s$ , the first order approximation of the eigenvalue perturbation  $\delta\pi$  is

$$\delta\pi = \frac{p^T (\mathbf{N}_s - \pi \mathbf{N}) q}{p^T \mathbb{L} q}$$

**A bound for pole sensitivity (Gosea).** It can be shown that:

$$\delta\pi = \mathbf{d}^T \epsilon, \text{ where } \epsilon = \begin{bmatrix} \epsilon_\mu \\ \epsilon_\lambda \end{bmatrix}, \text{ and } \mathbf{d} = \begin{bmatrix} \text{diag}(p)(\pi\mathbf{I} - \mathbf{M})\mathcal{V}\mathcal{C}_{\mu,\lambda}q \\ \text{diag}(q)(\pi\mathbf{I} - \mathbf{A})\mathcal{W}\mathcal{C}_{\lambda,\mu}p \end{bmatrix}.$$

The following sequence of inequalities holds

$$\begin{aligned} \|\mathbf{d}\|_2 &\leq \|\text{diag}(p)(\pi\mathbf{I} - \mathbf{M})\mathcal{V}\mathcal{C}_{\mu,\lambda}q\|_2 + \|\text{diag}(q)(\pi\mathbf{I} - \mathbf{A})\mathcal{W}\mathcal{C}_{\lambda,\mu}p\|_2 \\ &\leq \underbrace{\|p\|_\infty \max_i |\pi_i - \mu_i| \max_i |H(\mu_i)|}_{\leq \|p\|_2} \|\mathcal{C}_{\mu,\lambda}\|_2 \|q\|_2 + \underbrace{\|q\|_\infty \max_i |\pi_i - \lambda_i| \max_i |H(\lambda_i)|}_{\leq \|q\|_2} \|\mathcal{C}_{\lambda,\mu}\|_2 \|p\|_2 \\ &\leq \|p\|_2 \|q\|_2 \|\mathcal{C}_{\mu,\lambda}\|_2 \left[ \max_i |\pi - \mu_i| \max_i |H(\mu_i)| + \max_i |\pi - \lambda_i| \max_i |H(\lambda_i)| \right], \end{aligned}$$

since  $\|\mathcal{C}_{\mu,\lambda}\|_2 = \|\mathcal{C}_{\lambda,\mu}\|_2$ . Furthermore:

$$q_r = (\mathcal{C}_{\lambda,\pi}^T)^+ \mathbf{e}_r, \quad p_r = (\mathcal{C}_{\mu,\pi}^T)^+ \mathbf{e}_r.$$

Thus

$$\|\mathbf{d}_r\|_2 \leq \|(\mathcal{C}_{\mu,\pi}^T)^+ \mathbf{e}_r\|_2 \|(\mathcal{C}_{\lambda,\pi}^T)^+ \mathbf{e}_r\|_2 \|\mathcal{C}_{\mu,\lambda}\|_2 \mathbf{K}_{r,\mu,\lambda}$$

where  $\mathbf{K}_{r,\mu,\lambda} = \max_i |\pi_r - \mu_i| \max_i |H(\mu_i)| + \max_i |\pi_r - \lambda_i| \max_i |H(\lambda_i)|.$

Hence, the upper bound of  $|\delta\pi|$  corresponding to the perturbation of the pole  $\pi_r$  is a product of the following factors:

1. The absolute value of its corresponding residue  $\gamma_r$ .
2. the sensitivity (might be large depending on the selection/partition of sampling points).
3. the norm of the Cauchy matrix containing solely sampling points (could be large when the points are very close to each other).
4. A constant  $\mathbf{K}_{r,\mu,\lambda}$  that depends on the sampling points and on the magnitude of the original function evaluated and these points (not that large).

# Pseudospectra

- Consider  $\mathbf{A} \in \mathbb{C}^{n \times n}$ , its  $\epsilon$ -pseudospectrum is:

$$\Lambda_\epsilon(\mathbf{A}) = \left\{ z \in \mathbb{C} : \|(z\mathbf{I} - \mathbf{A})^{-1}\| \geq \epsilon^{-1} \right\}$$

$$\Updownarrow$$

$$\Lambda_\epsilon(\mathbf{A}) = \{ z \in \mathbb{C} : z \in \sigma(\mathbf{A} + \Delta\mathbf{A}) \text{ for some } \Delta\mathbf{A} \text{ with } \|\Delta\mathbf{A}\| \leq \epsilon \}$$

$$\Updownarrow$$

$$\Lambda_\epsilon(\mathbf{A}) = \{ z \in \mathbb{C} : \sigma_{\min}(z\mathbf{I} - \mathbf{A}) \leq \epsilon \}$$

- Given  $(\mathbf{A}, \mathbf{E}) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ , and  $\epsilon > 0$ , its  $\epsilon - (\gamma, \delta)$  pseudospectrum is:

$$\sigma_\epsilon^{(\gamma, \delta)} = \{ z \in \mathbb{C} \text{ is an eigenvalue of the pencil } z(\mathbf{E} + \Delta) - (\mathbf{A} + \Gamma) \\ \text{for some } \Gamma, \Delta \in \mathbb{C}^{n \times n} \text{ with } \|\Gamma\| \leq \epsilon\gamma, \|\Delta\| \leq \epsilon\delta \}$$

$$\Updownarrow$$

$$\sigma_\epsilon^{(\gamma, \delta)}(\mathbf{A}, \mathbf{E}) = \left\{ z \in \mathbb{C} : \|(z\mathbf{E} - \mathbf{A})^{-1}\| > \frac{1}{\epsilon(\gamma + |z|\delta)} \right\}$$

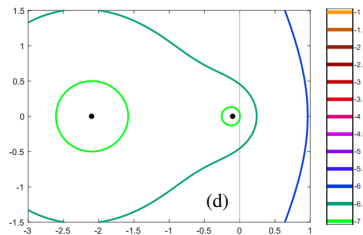
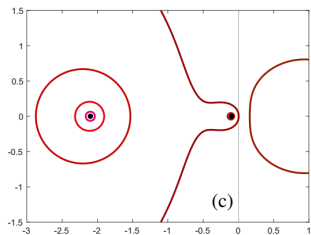
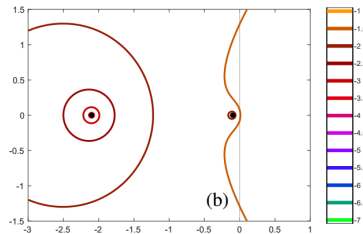
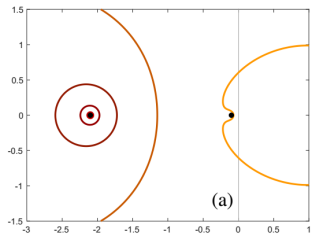
### Example 1 (Embree-Ionita)

Consider a system with realization

$$\mathbf{A} = \begin{bmatrix} -1.1 & 1 \\ 1 & -1.1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Sensitivities after taking measurements.

$\lambda_1$	$\lambda_2$	$\mu_1$	$\mu_2$	Sens ( $\pi_1 = -0.1$ )	Sens ( $\pi_2 = -2.1$ )
0.00	1.00	0.00+1.00i	0.00-1.00i	2.416e-01	2.430e+01
0.25	0.75	0.00+2.00i	0.00-2.00i	1.695e+00	2.306e+01
0.40	0.60	0.00+4.00i	0.00-4.00i	5.132e+00	3.221e+01
8.00	9.00	10.00	11.00	3.828e+05	6.828e+05



### Example 2 (Embree-Ionita)

Consider a system with realization

$$\mathbf{A} = \text{diag}(-1, -2, \dots, -10), \quad \mathbf{B} = [1, 1, \dots, 1]^T, \quad \mathbf{C} = [1, 1, \dots, 1].$$

The sensitivities of the poles after taking measurements are:

$\lambda$	$\mu$	Poles	Sensitivity	$\lambda$	$\mu$	Poles	Sensitivity
-10.25	-9.75	-1.000	4.781e-02	-5.25	-10.25	-10.000	4.781e-02
-9.25	-8.75	-10.000	4.781e-02	-4.75	-9.75	-1.000	4.781e-02
-8.25	-7.75	-9.000	4.951e-02	-4.25	-9.25	-2.000	4.951e-02
-7.25	-6.75	-2.000	4.951e-02	-3.75	-8.75	-9.000	4.951e-02
-6.25	-5.75	-3.000	4.989e-02	-3.25	-8.25	-8.000	4.989e-02
-5.25	-4.75	-8.000	4.989e-02	-2.75	-7.75	-3.000	4.989e-02
-4.25	-3.75	-7.000	5.004e-02	-2.25	-7.25	-4.000	5.004e-02
-3.25	-2.75	-4.000	5.004e-02	-1.75	-6.75	-7.000	5.004e-02
-2.25	-1.75	-5.000	5.009e-02	-1.25	-6.25	-6.000	5.009e-02
-1.25	-0.75	-6.000	5.009e-02	-0.75	-5.75	-5.000	5.009e-02

Setting	$\text{cond}(\mathbb{L}_S)$	$\text{cond}(\mathbb{L})$	$\text{cond}(\mathcal{C}_{\lambda, \pi})$	$\text{cond}(\mathcal{C}_{\mu, \pi})$
1	1.0814e+01	1.4685e+00	1.2172e+00	1.2172e+00
2	2.2385e+06	1.0389e+06	1.7707e+06	1.7707e+06



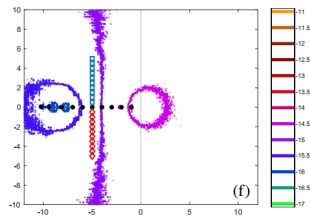
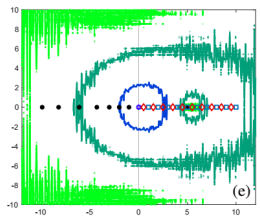
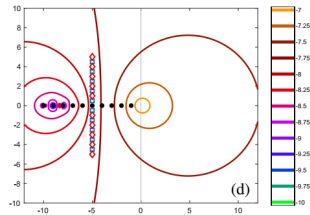
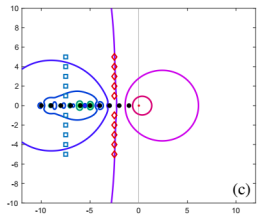
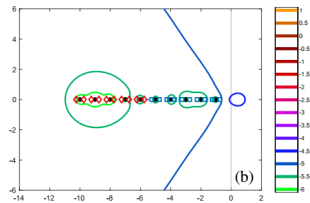
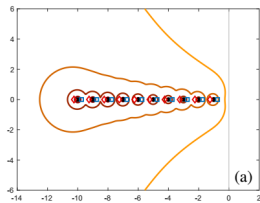
$\lambda$	$\mu$	Poles	Sensitivity	$\lambda$	$\mu$	Poles	Sensitivity
-5.00+0.50i	-5.00+1.00i	-5.000	1.375e+04	-5.00+0.50i	-5.00-0.50i	-5.000	1.676e+04
-5.00-0.50i	-5.00-1.00i	-4.000	1.721e+05	-5.00+1.00i	-5.00-1.00i	-4.000	1.923e+05
-5.00+1.50i	-5.00+2.00i	-6.000	3.873e+05	-5.00+1.50i	-5.00-1.50i	-5.999	5.598e+05
-5.00-1.50i	-5.00-2.00i	-3.000	2.544e+06	-5.00+2.00i	-5.00-2.00i	-2.998	2.703e+06
-5.00+2.50i	-5.00+3.00i	-1.000	3.549e+06	-5.00+2.50i	-5.00-2.50i	-0.999	3.620e+06
-5.00-2.50i	-5.00-3.00i	-2.000	8.533e+06	-5.00+3.00i	-5.00-3.00i	-1.996	8.864e+06
-5.00+3.50i	-5.00+4.00i	-7.000	1.385e+07	-5.00+3.50i	-5.00-3.50i	-6.972	2.778e+07
-5.00-3.50i	-5.00-4.00i	-10.000	7.884e+07	-5.00+4.00i	-5.00-4.00i	-9.942	3.203e+07
-5.00+4.50i	-5.00+5.00i	-8.000	1.365e+08	-5.00+4.50i	-5.00-4.50i	-8.722	1.607e+08
-5.00-4.50i	-5.00-5.00i	-9.000	2.875e+08	-5.00+5.00i	-5.00-5.00i	-7.769	2.048e+08

Setting	cond ( $\mathbb{L}_s$ )	cond ( $\mathbb{L}$ )	cond ( $\mathcal{C}_{\lambda,s}$ )	cond ( $\mathcal{C}_{\mu,s}$ )
4	2.5548e+08	4.1502e+08	1.8096e+04	2.4538e+04
6	8.4840e+15	1.2067e+16	1.0403e+08	1.0403e+08

$\lambda$	$\mu$	Poles	Sensitivity	$\lambda$	$\mu$	Poles	Sensitivity
-2.50+1.00i	-7.50+1.00i	-1.000	3.836e+03	1.00	0.50	-1.000	2.745e+15
-2.50-1.00i	-7.50-1.00i	-2.000	1.553e+04	2.00	1.50	1.999	6.323e+15
-2.50+2.00i	-7.50+2.00i	-10.000	5.356e+04	3.00	2.50	-2.000	5.334e+17
-2.50-2.00i	-7.50-2.00i	-3.000	5.755e+04	4.00	3.50	-9.839	2.667e+18
-2.50+3.00i	-7.50+3.00i	-8.000	1.741e+05	5.00	4.50	-3.017	9.103e+18
-2.50-3.00i	-7.50-3.00i	-9.000	2.093e+05	6.00	5.50	-7.974	1.410e+19
-2.50+4.00i	-7.50+4.00i	-7.000	2.383e+05	7.00	6.50	-5.888	2.653e+19
-2.50-4.00i	-7.50-4.00i	-4.000	3.306e+05	8.00	7.50	-4.216	2.654e+19
-2.50+5.00i	-7.50+5.00i	-6.000	6.402e+05	9.00	8.50	1.054	7.516e+19
-2.50-5.00i	-7.50-5.00i	-5.000	7.879e+05	10.00	9.50	9.143	Inf

Setting	cond ( $\mathbb{L}_s$ )	cond ( $\mathbb{L}$ )	cond ( $\mathcal{C}_{\lambda,s}$ )	cond ( $\mathcal{C}_{\mu,s}$ )
3	4.2868e+09	4.1407e+09	2.3458e+06	3.4688e+05
5	4.5784e+16	2.3000e+16	2.9699e+12	4.6013e+09

$\lambda$	$\mu$	Poles	Sensitivity
1.00	1.00	-1.000	6.644e+11
2.00	2.00	-2.001	3.175e+14
3.00	3.00	-9.854	8.299e+15
4.00	4.00	-3.028	8.781e+15
5.00	5.00	-4.270	3.503e+16
6.00	6.00	-8.042	3.806e+16
7.00	7.00	-5.977	5.223e+16
8.00	8.00	2.164	6.988e+17
9.00	9.00	2.995	5.052e+18
10.00	10.00	2.825	8.028e+18



### Beam Example (SISO, order = 348)

The different setting of interpolation points are as follows.

Let  $\omega = \text{logspace}(-2, 1, 500)$ ,  $\omega_1 = \text{logspace}(-2, 1, 200)$ . Then

- Setting 1 (interlaced):

$$\lambda = \{-i\omega(1 : 2 : 500)\} \cup \{i\omega(1 : 2 : 500)\}, \quad \mu = \{-i\omega(2 : 2 : 500)\} \cup \{i\omega(2 : 2 : 500)\}$$

- Setting 2 (split):  $\lambda = \{-i\omega\}$ ,  $\mu = \{i\omega\}$ .

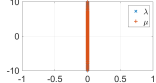
- Setting 3 (interlaced and shifted) is obtained from setting 1 by shifting the points:

$$\begin{aligned} \lambda &= \{3 - i\omega(1 : 2 : 500)\} \cup \{3 + i\omega(1 : 2 : 500)\}, \\ \mu &= \{3 - i\omega(2 : 2 : 500)\} \cup \{3 + i\omega(2 : 2 : 500)\}. \end{aligned}$$

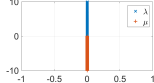
- Setting 4 (interlaced obtained from  $\omega_1$ ):

$$\lambda = \{-i\omega_1(1 : 2 : 200)\} \cup \{i\omega_1(1 : 2 : 200)\}, \quad \mu = \{-i\omega_1(2 : 2 : 200)\} \cup \{i\omega_1(2 : 2 : 200)\}$$

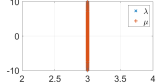
Interpolation Points (Setting 1)



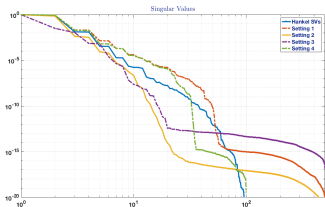
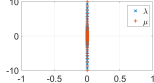
Interpolation Points (Setting 2)



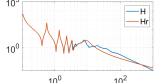
Interpolation Points (Setting 3)



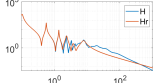
Interpolation Points (Setting 4)



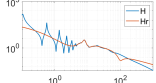
Frequency Response (Setting 1)



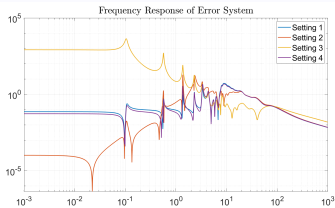
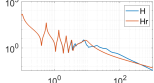
Frequency Response (Setting 2)



Frequency Response (Setting 3)



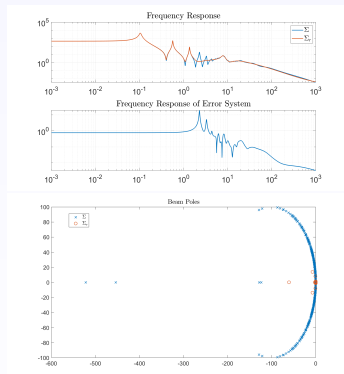
Frequency Response (Setting 4)



Poles	Sensitivity	Poles	Sensitivity
-0.005-0.105i	2.166e-03	-0.005+0.105i	6.170e-03
-0.005+0.105i	2.166e-03	-0.005-0.105i	6.170e-03
-0.007-0.569i	8.215e-03	-0.007-0.569i	4.988e+00
-0.007+0.569i	8.215e-03	-0.007+0.569i	4.988e+00
-0.014-1.369i	1.264e-01	-0.014-1.372i	5.449e+02
-0.014+1.369i	1.264e-01	-0.014+1.372i	5.449e+02
-0.032+2.305i	1.546e+00	-0.091-2.361i	1.118e+04
-0.032-2.305i	1.546e+00	-0.091+2.361i	1.118e+04
-0.066+3.352i	1.269e+01	-0.919+7.723i	6.678e+04
-0.066-3.352i	1.269e+01	-0.919-7.723i	6.678e+04
-1.705-9.222i	9.223e+02	-0.255-3.751i	1.167e+05
-1.705+9.222i	9.223e+02	-0.255+3.751i	1.167e+05

Poles	Sensitivity	Poles	Sensitivity
0.059-0.000i	1.734e+05	-0.005-0.105i	4.748e-03
-0.660-7.986i	2.105e+05	-0.005+0.105i	4.748e-03
-0.660+7.986i	2.105e+05	-0.007-0.569i	2.691e-02
-0.423+0.000i	7.663e+05	-0.007+0.569i	2.691e-02
-0.823-5.777i	2.545e+06	-0.014+1.369i	9.268e-01
-0.823+5.777i	2.545e+06	-0.014-1.369i	9.268e-01
1.556-3.498i	1.234e+07	-0.032-2.305i	5.709e+00
1.556+3.498i	1.234e+07	-0.032+2.305i	5.709e+00
-7.036-13.182i	2.125e+07	-0.072+3.379i	4.479e+01
-7.036+13.182i	2.125e+07	-0.072-3.379i	4.479e+01
-35.345+62.830i	4.317e+09	-1.696-9.016i	1.545e+03
-35.345-62.830i	4.317e+09	-1.696+9.016i	1.545e+03

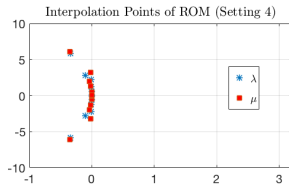
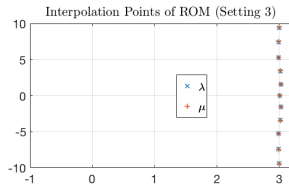
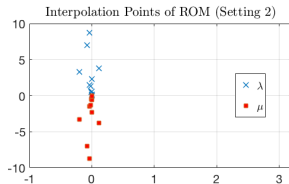
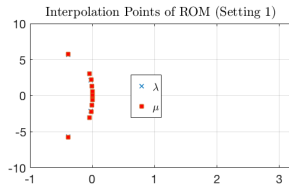
## IRKA



Poles	Sensitivity
-0.005-0.105i	1.049e-02
-0.005+0.105i	1.049e-02
-0.007+0.569i	1.839e-02
-0.007-0.569i	1.839e-02
-0.014-1.368i	1.274e-01
-0.014+1.368i	1.274e-01
-0.719+8.004i	3.838e+01
-0.719-8.004i	3.838e+01
-2.221-0.000i	8.695e+02
-7.269-13.763i	3.288e+03
-7.269+13.763i	3.288e+03
-60.652+0.000i	3.987e+04

Poles and Sensitivities of ROM constructed by IRKA

## Loewner System Constructed by Interpolation Points of ROMs





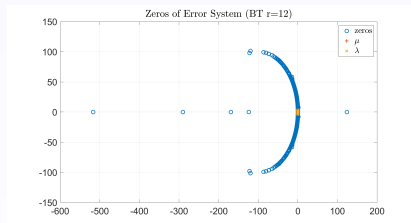
Poles	Sensitivity	Poles	Sensitivity
-0.007+0.569i	1.577e+02	-0.005-0.105i	2.661e+02
-0.007-0.569i	1.697e+02	-0.005+0.105i	8.170e+02
-0.005+0.105i	2.449e+02	-0.007-0.569i	2.314e+05
-0.005-0.105i	3.061e+02	-0.007+0.569i	2.434e+05
-0.014+1.369i	1.378e+03	-0.014-1.372i	8.155e+06
-0.014-1.369i	1.423e+03	-0.014+1.372i	1.797e+07
-0.032+2.305i	1.137e+04	-0.091-2.361i	3.302e+07
-0.032-2.305i	1.151e+04	-0.091+2.361i	5.944e+07
-0.066+3.352i	1.319e+05	-0.919-7.723i	1.263e+08
-0.066-3.352i	1.322e+05	-0.919+7.723i	1.383e+08
-1.705-9.222i	3.850e+06	-0.255-3.751i	3.079e+08
-1.705+9.222i	3.860e+06	-0.255+3.751i	3.512e+08

Poles	Sensitivity	Poles	Sensitivity
0.059-0.000i	5.065e+12	-0.007-0.569i	1.779e-01
-0.660+7.986i	5.180e+12	-0.005-0.105i	2.215e-01
-0.660-7.986i	5.180e+12	-0.005+0.105i	2.336e-01
-0.423+0.000i	2.113e+13	-0.007+0.569i	2.354e-01
-0.823+5.777i	6.286e+13	-0.014+1.369i	7.310e+00
-0.823-5.777i	6.286e+13	-0.014-1.369i	8.774e+00
-7.036+13.182i	1.889e+14	-0.032+2.305i	1.964e+01
-7.036-13.182i	1.889e+14	-0.032-2.305i	2.175e+01
1.556+3.498i	3.739e+14	-0.072-3.379i	6.555e+02
1.556-3.498i	3.739e+14	-0.072+3.379i	6.648e+02
-35.345+62.830i	2.973e+15	-1.696-9.016i	1.187e+04
-35.345-62.830i	2.973e+15	-1.696+9.016i	1.223e+04

Poles and Sensitivities of Loewner system, setting 1-4 from left to right

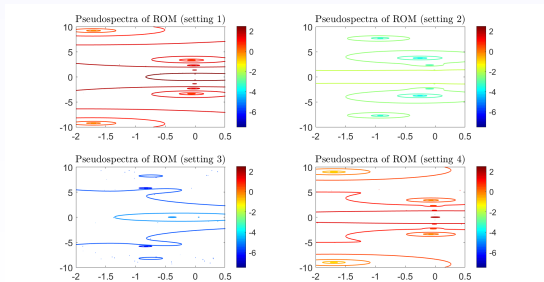
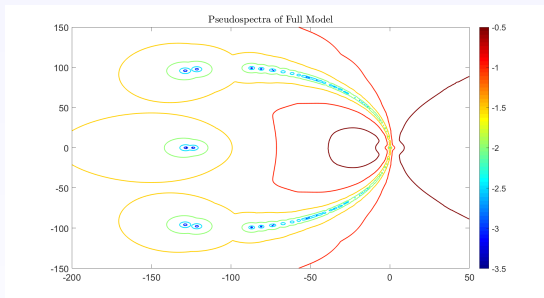
## Balanced Truncation $k = 12$

Obtain ROM by balanced truncation. Then choose the zeros of error system as interpolation points to construct the Loewner system.



Poles	Sensitivity
-0.035-2.297i	1.388e+00
-0.015-1.368i	2.269e+00
-0.015+1.368i	3.443e+00
-0.007-0.569i	4.357e+00
-0.007+0.569i	4.362e+00
-0.035+2.297i	4.462e+00
-0.005-0.105i	7.556e+00
-0.005+0.105i	8.146e+00
-0.092-3.278i	7.115e+01
-0.092+3.278i	7.507e+02
-1.043-7.469i	1.290e+05
-1.043+7.469i	3.966e+05

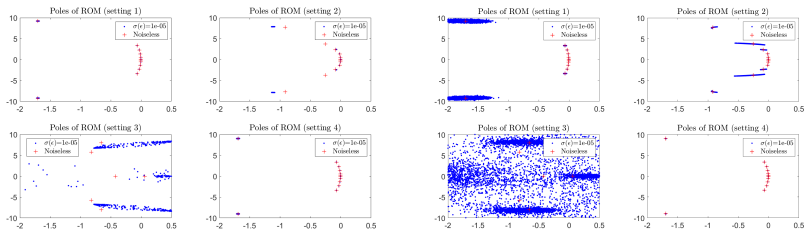
# Pseudospectra



## Pole Distribution of ROMs with perturbation on the measurements.

Below we show the poles of 10000 ROMs with perturbation on the measurements.

The first figure shows the pole distribution of ROMs with perturbation on the original measurements. The second figure shows the pole distribution of ROMs with perturbation on the ROM's interpolation points.



# References

1. A.C. ANTOUNAS, *The Loewner Framework and Transfer Functions of Singular/Rectangular Systems*, Applied Mathematics Letters, **54**: 36-47 (2016).
2. M. EMBREE AND A.C. IONITA, *Pseudospectra of Loewner matrix pencils*, arXiv preprint, arXiv: 1910.12153 (2019).
3. B. BECKERMAN, G.H. GOLUB, AND G. LABAHN, *On the numerical condition of a generalized Hankel eigenvalue problem*, Numer. Math. **106**: 41-68 (2007).
4. A.C. ANTOUNAS, S. LEFTERIU AND A.C. IONITA, *A tutorial introduction to the Loewner Framework for Model Reduction*, in Model Reduction and Approximation: Theory and Algorithms, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia, Computational Science and Engineering, volume CS15, pages 335-375, 2017.
5. C.A. BEATTIE AND S. GUGERCIN, *Model Reduction by Rational Interpolation*, in Model Reduction and Approximation: Theory and Algorithms, Edited by P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, SIAM, Philadelphia, Computational Science and Engineering, volume CS15, pages 297-334, 2017.
6. B. BECKERMAN, A. TOWNSEND, *On the Singular Values of Matrices with Displacement Structure*, SIAM J. Matrix Anal. Appl., Vol. 38, No. 4, 1227-1248, 2017.
7. G.H. GOLUB, P. MILANFAR, AND J. VARAH, *A stable numerical method for inverting shape from moments*, SIAM J. Sci. Comput., Vol. 21, No. 4, pp. 1222-1243 (1999).

8.

